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ON THE BOUNDARY CONDITIONS FOR NAVIER–STOKES EQUATIONS IN STREAM FUNCTION–VORTICITY VARIABLES IN SIMULATION OF A FLOW AROUND A SYSTEM OF BODIES

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A method of determining the boundary conditions for the Navier–Stokes equations in stream function–vorticity variables, used for simulation of a nonstationary, asymmetric laminar flow of an incompressible viscous fluid around bodies, has been proposed. Universal relations for desired functions on surfaces around which the stream flows, independent of the method of spatial discretization, have been obtained.

Introduction. It is customary to use the Navier–Stokes equations in stream function ψ–vorticity ω variables for simulation of plane laminar flows of an incompressible viscous fluid [1–3] because these equations are simpler from the standpoint of numerical realization than the analogous equations in natural velocity u, v–pressure p variables. However, as is known, it is difficult to write the boundary conditions for the ψ–ω model, which limits the use of it for calculating nonstationary, asymmetric viscous flows. The point is that for the equation of vorticity transfer

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}} \Delta \omega , \quad x, y \in D \tag{1}$$

there are no physical boundary conditions on the surfaces γ_i (i = 1, 2, ...), around which a stream flows, while for the Poisson equation

$$\Delta \psi = - \omega , \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} , \quad u = \frac{\partial \psi}{\partial y} , \quad v = - \frac{\partial \psi}{\partial x} , \tag{2}$$

determining the stream function there exist two natural boundary conditions:

$$x, y \in \gamma_i : \quad \psi = \psi_i , \quad \frac{\partial \psi}{\partial n} = v_i . \tag{3}$$

Hereinafter, n denotes the outer normal to the boundary and v_i denotes the tangential velocity of the fluid on γ_i. The quantities ψ_i are constants in the case where the adhesion conditions are set on the surfaces around which the fluid flows and the equality

$$\psi_i (s) = \psi_i^0 + \int_0^s v_n^i (s) \, ds$$

is fulfilled when the fluid is blown in or drawn off with a velocity v_n^i. In this equation, integration is performed over the arcs of the surface γ_i from the arbitrary point $s = 0$ at which the constant ψ_i^0 should be determined.

Traditionally [1, 2], the first equality of (3) is considered as the Dirichlet boundary condition for Eq. (2), and the second equation — the Neumann condition — is used for determining the vorticity at the boundary ω_γ. This prob-

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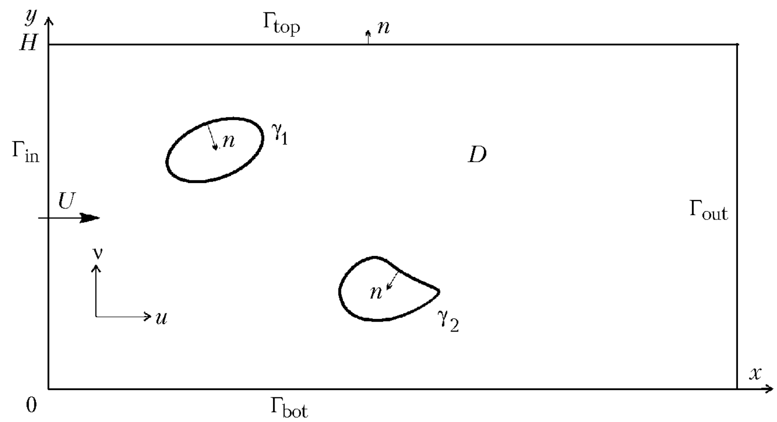

Fig. 1. Scheme of a uniform flow around a system of bodies.

lem was solved in a large number of works, of which review [4] stands out. The authors of this work thoroughly analyzed different variants of finite difference and simpler finite-element representations of the Thom–Burgraf formula and its higher-order analogs. Moreover, they proposed an integral condition for ω_γ that suggests the introduction of a special system of basis functions at the boundary and leads, on discretization, to a system of equations with a filled asymmetric matrix. However, this system, in our opinion, is not fully suitable for practical calculations.

Along with the determination of the boundary conditions for Eq. (1), of crucial importance is the prescription of the quantities ψ_i and v_i in relations (3). So far, in solving theoretical problems with the use of the ψ–ω model, the stream function and velocity at a wall were assumed to be definite in this model. However, in practice these quantities can be definite only in certain cases, e.g., in a flow around a channel or in a steady-state symmetric flow in an individual body.

In the present work, we considered the fairly common problem on a viscous-fluid flow around a system of bodies. The boundary conditions for the Navier–Stokes equations in transformed variables have been formulated, and numerical methods of solving these equations have been proposed.

Formulation of the Problem. We considered the model problem on a plane viscous-fluid flow around a system of N bodies (see Fig. 1). The flow region D is bounded by the input cross section Γ_{in}, for which we constructed a velocity diagram with a flow rate Q:

$$u = U(y), \quad v = 0; \quad \psi = \psi_0(y) = \int_0^y U(\xi)\, d\xi, \quad \psi_0(H) = Q;$$

streamlines Γ_{top} and Γ_{bot}; a distant output cross section Γ_{out}, in which $v = 0$ is assumed to be equal to zero, and impenetrable inner surfaces γ_i ($i = 1, ..., N$) of the bodies around which the stream flows, at which the boundary conditions (3) are set.

Two different formulations of the problem were considered within the framework of the above scheme.

1. The horizontal streamlines Γ_{top} and Γ_{bot} are immovable walls of a channel, at which the adhesion conditions $v_n = v_\tau = 0$ are set.

2. The streamlines Γ_{top} and Γ_{bot} separate a periodicity cell in a grid of the surfaces around which the stream flows. In this case, the symmetry (ideal slipping) conditions $v_n = 0$, $\partial v_\tau/\partial n = 0$ are set and the tangential components of the velocity vector at the upper and lower boundaries are assumed to be equal.

The external boundary conditions for Eqs. (1) and (2) have, in accordance with the above-described scheme of flow, the following form:

$$x, y \in \Gamma_{in}: \quad \psi = \psi_0, \quad \frac{\partial \psi}{\partial n} = 0, \quad \omega = \omega_0 \equiv -\frac{du}{dy}; \tag{4}$$

$$x, y \in \Gamma_{\text{out}}: \ \frac{\partial \psi}{\partial n} = 0 \,, \ \ \frac{\partial \omega}{\partial n} = 0 \,; \tag{5}$$

$$x, y \in \Gamma_{\text{top,bot}}: \ \psi = \psi_0 (y) \,, \ \ \frac{\partial \psi}{\partial n} = 0 \,; \tag{6}$$

$$x, y \in \Gamma_{\text{top,bot}}: \ \psi = \psi_0 (y) \,, \ \ \omega = 0 \,. \tag{7}$$

Boundary condition (6) was written for problem 1 and boundary condition (7) was written for problem 2. In addition, the initial vorticity distribution should be determined; it can be, e.g., $\omega = 0$.

Note that the velocities v_i in boundary conditions (3) are assumed to be definite (more exactly they are determined using additional models of interaction of the flow with the bodies) and the stream function ψ_i at the boundaries is calculated by relations (1)–(7).

Determination of the Stream Function at the Boundaries of Bodies around Which a Stream Flows. We first consider problem 2, for which the ideal-slipping and periodicity conditions are set on the horizontal walls. Let us introduce the function $\eta_1(x, y)$ satisfying the boundary-value problem

$$\Delta \eta_1 = 0 \,, \ \ x, y \in D \,; \ \ \Gamma_{\text{in,out}} \,, \ \ \gamma_{2,..,N}: \ \frac{\partial \eta_1}{\partial n} = 0 \,; \ \ \Gamma_{\text{top,bot}}: \ \eta_1 = 0 \,; \ \ \gamma_1: \ \frac{\partial \eta_1}{\partial n} = 1 \,. \tag{8}$$

Multiplying Eq. (8) by ψ and integrating it over the region D, we obtain, using the Green formula,

$$\int_D \nabla \psi \cdot \nabla \eta_1 dD = \int_\Gamma \psi \frac{\partial \eta_1}{\partial n} ds \,. \tag{9}$$

Here Γ denotes integration over all boundaries. Multiplying Eq. (2) by the function η_1 and integrating it over D, we obtain

$$\int_D \Delta \psi \eta_1 dD = \int_\Gamma \eta_1 \frac{\partial \psi}{\partial n} ds - \int_D \nabla \psi \cdot \nabla \eta_1 dD = - \int_D \omega \eta_1 dD \,.$$

Substitution of formula (9) into this equality gives the integral relation

$$\int_\Gamma \eta_1 \frac{\partial \psi}{\partial n} ds - \int_\Gamma \psi \frac{\partial \eta_1}{\partial n} ds = - \int_D \omega \eta_1 dD \,, \tag{10}$$

which should be true for any harmonic function η_1. Identity (10) was written apparently for the first time in [5] as the condition of consistency between the vorticity field and the stream function in the Navier–Stokes equations and was used for formulation of nonlocal boundary conditions for ω on γ. Below, the system of relations (10) will be used for determining the constants ψ_i.

It follows from boundary conditions (4), (5), (7), and (8) that

$$\int_\Gamma \eta_1 \frac{\partial \psi}{\partial n} ds = \sum_{i=1}^{N} v_i \int_{\gamma_i} \eta_1 ds \,, \ \ \int_\Gamma \psi \frac{\partial \eta_1}{\partial n} ds = \psi_1 \, |\gamma_1| + Q \int_{\Gamma_{\text{top}}} \frac{\partial \eta_1}{\partial n} ds \,.$$

Substitution of these integrals over the boundary into (10) gives an explicit expression for the boundary value of the stream function ψ_1 expressed in terms of the vorticity ω and the auxiliary function η_1 — a solution of auxiliary problem (8):

$$\psi_1 = \frac{1}{|\gamma_1|}\left(\sum_{i=1}^{N} v_i \int_{\gamma_i} \eta_1 ds - Q \int_{\Gamma_{\text{top}}} \frac{\partial \eta_1}{\partial n} ds + \int_D \eta_1 \omega dD \right). \tag{11}$$

The constants ψ_i (i = 2, ..., N) are determined analogously using the auxiliary problems

$$\Delta \eta_i = 0\,, \quad x, y \in D\,; \quad \Gamma_{\text{in,out}}\,, \quad \gamma_{k \neq i} : \quad \frac{\partial \eta_i}{\partial n} = 0\,; \quad \Gamma_{\text{top,bot}} : \quad \eta_i = 0\,; \quad \gamma_i : \quad \frac{\partial \eta_i}{\partial n} = 1\,. \tag{12}$$

Thus, to determine the dynamics of the stream function $\psi_i(t)$ on the surfaces γ_i, it will suffice to preliminarily solve auxiliary problems (8) and (12) and then to use formula (11) at each time step in the process of solving non-stationary problem (1), (2).

In the case of simulation of a symmetric flow around an individual body, expression (11) can take a simpler form, the trial function η_1 is symmetric relative to the middle line $y = H/2$, and the equality

$$\int_{\Gamma_{\text{top}}} \frac{\partial \eta_1}{\partial n} ds = -\frac{|\gamma_1|}{2}$$

follows from (8). Moreover, in the periodic problem considered, the condition of solution of Eq. (2) is

$$v_1 \, |\gamma_1| = -\int_D \omega dD\,.$$

Substituting these expressions into (11), we obtain the descriptive formula

$$\psi_1 = \frac{Q}{2} + \frac{1}{|\gamma_1|} \int_D \omega\,(\eta_1 - \bar{\eta})\, dD\,, \quad \bar{\eta} \equiv \frac{1}{|\gamma_1|} \int_{\gamma_1} \eta_1 ds\,, \tag{13}$$

which, at $v_1 = 0$, gives the trivial result $\psi_1 = Q/2$.

We now consider problem 1, in which adhesion condition (6) is set on the channel walls. In this case, the auxiliary functions η_i are conveniently determined in the following formulation:

$$\Delta \eta_i = 0\,, \quad x, y \in D\,; \quad \Gamma_{\text{in,out,top}}\,, \quad \gamma_{k \neq i} : \quad \frac{\partial \eta_i}{\partial n} = 0\,; \quad \Gamma_{\text{bot}} : \quad \eta_i = 0\,; \quad \gamma_i : \quad \frac{\partial \eta_i}{\partial n} = 1\,, \tag{14}$$

and formulas (11) take a simpler form:

$$\psi_i = \frac{1}{|\gamma_i|}\left(\sum_{k=1}^{N} v_k \int_{\gamma_k} \eta_i ds + \int_D \eta_i \omega dD \right), \quad i = 1, 2, ..., N\,. \tag{15}$$

The method proposed for determining the boundary values of the stream function will be used without changes in the case of a channel with curvilinear boundaries.

Calculation of the Boundary Values of the Vorticity. If an algorithm based on separate, successive solving of Eqs. (1) and (2) is used for numerical solution, it is best to set main boundary conditions for ω rather than the Neumann conditions for the stream function (3). They are usually derived [1–4, 6–8] by discrete representation of Eq. (2) so that the boundary conditions $\partial\psi/\partial n\,|_\gamma = v$ are fulfilled. However, the use of special methods of approximation in the neighborhood of the boundaries can lead to the appearance of nonphysical sources of vorticity and, as a consequence, to a distortion of the actual pattern of the flow. Below, we propose a universal approach to the construction of grids, consistent with the main computational scheme, for approximations of the boundary conditions for a vortex.

Equation (2) for the stream function is solved using only the Dirichlet condition from (3), (6) and (7), and the values of ψ at the boundaries around which the stream flows are determined from formulas (11) and (15). The grid approximation is constructed on the basis of the integral identity

$$\int_D \nabla\psi \cdot \nabla\eta dD = \int_D \omega\eta dD\,, \tag{16}$$

in which η is an arbitrary function equal to zero at the boundary points, at which the Dirichlet conditions are set, and $\psi = \psi_i$ at the boundaries γ_i. Equation (16) leads, on finite-element or finite-difference discretization, to a system of linear algebraic equations. The main boundary conditions are determined at the matrix level: the lines corresponding to the points with conditions of the first kind (the grid points lying on the boundaries γ_i, Γ_{in}, Γ_{top} and Γ_{bot}) are reduced to zero, the diagonal elements are assumed to be equal to unity, and definite values of ψ_i are written on the right side; symmetric charges are made in the columns.

Equation (2) with boundary conditions of the second kind is used for determining ω at the boundaries. In this case, the integral identity will have the form

$$\int_D \omega\eta dD = \int_D \nabla\psi \cdot \nabla\eta dD - \sum_{i=1}^{N} \int_{\gamma_i} v_i \eta ds\,, \tag{17}$$

where η is not limited by any boundary conditions. The change from (17) to finite-dimensional schemes is made by analogy with that for Eq. (16). The approximating equation has the form

$$\mathbf{M}\omega = \Lambda\psi + \mathbf{f}\,,$$

where $\mathbf{M}$ is a symmetric sparse matrix (element mass matrix in the terminology of the finite-element method), Λ is a grid analog of the Laplace operator with Neumann boundary conditions (fluid mass matrix), ψ is a solution of problem (16) satisfying the main boundary conditions, and $\mathbf{f}$ is the force vector that is due to the last term on the right side of (17) and differs from zero only at the movable boundaries. The deficient Dirichlet conditions for the vorticity are obtained by selecting ω values corresponding to the boundary points from the solution of the system of equations obtained. Note that the matrix $\mathbf{M}$ has a condition number of the order of unity; therefore, the system is conveniently solved using iteration methods, in particular the conjugate-gradient method with diagonal prestipulation [9].

Some points need to be made to the algorithm proposed. First, it is universal and can be used, without changes, in both finite-difference methods of discretization of Eq. (16) and in any finite-element methods of arbitrary accuracy. This algorithm is also insensitive to the shape of the flow-region boundary. Second, the boundary conditions for ω determined by this method have an approximation order equal to that of the grid schemes for Eqs. (1) and (2). Third, the boundary condition constructed on the basis of Eq. (17) does not disturb the conservatism of the grid schemes. In particular, a vorticity balance is obtained if $\eta \equiv 1$ in (17). Finally, it should be noted that, in some cases, the matrix $\mathbf{M}$ can be diagonal. For example, this is the case for finite-difference methods of the second order of approximation on an orthogonal grid and simpler finite-element methods on triangular and quadrangular grids, while the mass matrix is calculated using quadrature formulas by the tops of elements. Simple calculations have shown that, in the case where a uniform rectangular grid is used, Eq. (17) for a rectilinear boundary leads to the known Thom formula that is usually called the first-order formula [1–4].

On the Velocity of a Flow around Surfaces. In the above-considered problems, we used the quantities v_i determining the velocity of a fluid flow around bodies. In the simplest case, they represent definite time functions, using which we obtain, if adhesion conditions are set at the fluid-wall contact, a model of the action of moving surfaces on a viscous-fluid flow. The most well-known problems of this class are problems on a flow in a cavity with a movable cover and a flow around a rotating circular cylinder [8, 10, 11]. At the same time, the above-indicated mathematical formulas can be used in solving problems on interaction of a flow with bodies around which the stream flows in the case where the flow velocity at the walls is unknown.

By way of example, we will consider a flow around a circular cylinder with a fixed axis and a rotational degree of freedom.

If the cylinder has no inertia, its movement is completely determined by the flow; in this case, the body as a whole is free from shear stresses:

$$\frac{1}{\mathrm{Re}}\int_\gamma \sigma_\tau ds \sim \int_\gamma \omega ds = 0 \,. \tag{18}$$

Since the vorticity at the boundary, according to (17), depends linearly on v, Eq. (18) uniquely determines the velocity of rotation of the cylinder at each instant of time.

If the cylinder of radius R is made of a material with a density ρ_c and has a mass $m = \rho_c \pi R^2$ (per meter of the length), its rotation is determined by the equation [12]

$$M = FR\,, \quad F = \int_0^{2\pi R} \rho\nu \frac{\partial v_\tau}{\partial n} ds\,, \quad M = J\frac{d}{dt}\left(\frac{v_1}{R}\right), \quad J = \frac{mR^2}{2}\,. \tag{19}$$

From (19) follows the expression for the linear velocity v:

$$\frac{dv_1}{dt} = \frac{2\nu\rho}{\pi\rho_c R^2}\int_0^{2\pi R} \frac{\partial v_\tau}{\partial n} ds\,,$$

which can be written in the dimensionless form

$$\frac{dv_1}{dt} = -\frac{2}{\pi}\frac{K_\rho}{\mathrm{Re}}\int_\gamma \omega ds\,, \quad K_\rho = \frac{\rho}{\rho_c}\,. \tag{20}$$

Equation (20) should be supplemented with the initial conditions $v_1(0) = v_1^0$.

Note that, in the case of an infinitely light cylinder ($K_\rho \to \infty$), (20) is transformed into the condition of free rotation (18).

Algorithm of Numerical Solution. The approaches to the formulation of boundary conditions proposed in the present work were used for construction of algorithms for numerical solution of the problem on an external flow around a circular cylinder at Reynolds numbers of the order of 150. We considered the cases where the cylinder was

a) in the stationary state,

b) in the state of forced rotation with a definite velocity,

c) in the state of free rotation as a result of the interaction with the flow,

d) in the state of inertial rotation.

We first constructed a grid and solved problem (8) for determining the auxiliary function η_1. The initial conditions were set and then the functions ψ and ω were calculated in a time cycle $t_j = j\tau$ (j = 1, 2, ...).

The scheme of solving the problem in a time layer was as follows.

1. The rotational velocity of the cylinder is determined by the values of the functions ψ and ω in the previous time layer. The value of v_1 is definite in cases (a) and (b); in case (c), v_1 in Eq. (17) is selected such that condition (18) was fulfilled; in the case of inertial rotation (d), v_1 is determined using the explicit scheme for problem (20).

2. The problems for the stream function and vorticity are solved successively in an iteration cycle. At first, the boundary value of ψ_1 is determined by formula (13) and problem (16) is solved for ψ. The stream-function distribution obtained and the rotational velocity v_1 are used for calculating the vortex at the boundary in the process of solving problem (17). Then the Dirichlet problem is solved for Eq. (1), which on discretization with respect to time, takes the form

$$\omega - \frac{\tau}{\mathrm{Re}}\,\Delta\omega = \phi\,(\psi^{j-1}, \omega^{j-1})\,, \quad \phi\,(\psi, \omega) = \omega - \tau \left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} \right).$$

Spatial discretization of the problem for ω, as well as the other grid approximations, is done on the basis of the integral identity

$$\int_D \omega\eta dD + \frac{\tau}{\mathrm{Re}} \int_D \nabla\omega\cdot\nabla\eta dD = \int_D \phi\eta dD\,,$$

$\omega = \omega_\gamma$ at the boundary γ. The iterations are completed when

$$\max\,|\,\omega_\gamma^{k+1} - \omega_\gamma^{k}\,| \,/\max\,|\,\omega_\gamma^{k}\,| < \varepsilon \sim 10^{-4}\,,$$

where k is the number of an iteration.

This algorithm of numerically solving the Navier–Stokes problem was realized by the finite-element method with the use of nonuniform grids of linear triangular elements and successfully tested on the basis of known results of numerical simulation of a separation flow around an immovable circular cylinder and a circular cylinder rotating with a constant velocity [10, 11]. The results of the calculations will be presented and analyzed in a separate work.

Conclusions. One of the main results of the present work is the derivation of closing equations (11) and (13), which allow one to represent the stream function on the surface of bodies around which a complex stream flows in terms of auxiliary harmonic functions η_i. These functions represent solutions of the simple, one-type problems (12) and (14). They are determined only by the geometry of the computational region and, therefore, are calculated once. Another important result is Eq. (17) for the vorticity distribution over the surface of a body, consistent with the fields of ψ and η inside the computational region. In comparison with the traditional approach, where the values of ψ_i are determined based on certain geometric grounds and special formulas are written for ω at the boundary, the universal relations obtained by us practically do not increase the computational expenses. These relations allow one to substantially extend the range of problems on viscous flows that can be effectively solved within the framework of the Navier–Stokes model in transformed variables.

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NOTATION

D, computational region; **f**, force vector; F, viscous-friction force, H; J, moment of inertia, kg·m^2; **M**, element mass matrix; m, mass of the body, kg; N, number of bodies; n, normal to the boundary; p, pressure; Q, flow rate; R, radius of the cylinder, m; Re, Reynolds number; s, arc of the surface; t, t_j (j = 1, 2, ...), time, time layer of the computational scheme; u, v, v_n, v_τ, components of the velocity vector; U, velocity of the flow at the input to the channel; x, y, Cartesian coordinates; ε, criterion of completion of iterations; ϕ, right side of the equation for ω in a time layer; γ, γ_i, and γ_k (i, k = 1, 2, ..., N), surfaces of the bodies around which the stream flows; Γ, Γ_{top}, Γ_{bot}, Γ_{in}, Γ_{out}, outer boundaries of the computational region; η, trial function; η_i (i = 1, 2, ..., N), auxiliary harmonic function; Λ, stiffness matrix; ν, kinematic viscosity of the fluid, m^2/sec; ρ and ρ_c, densities of the fluid and the cylinder, kg/m^3; σ_τ, shear stress at the boundary; τ, time step of the computational scheme; ω, vorticity; ω_i and ω_γ, boundary values of ω; ψ, stream function; ψ_i, boundary values of ψ on the surface i. Subscripts: c, cylinder; i, k, ordinal number of a body; j, number of a time layer of the computational scheme; n, normal component; top, top; bot, bottom; in, input; out, output; γ, boundary value; ρ, dimensionless density criterion; τ, tangential component.

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